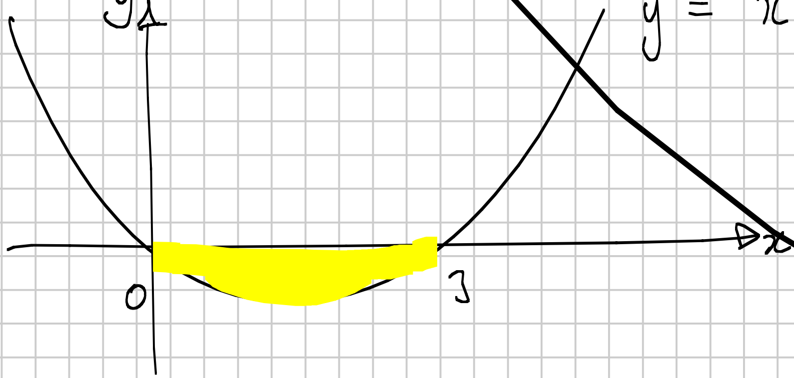
Note Title 28/05/2010

C2 chapter 11 integration part two

11.3 Areas under the x-axis



$y = x(x-3)$

$$\int_0^3 x(x-3)\,dx = \int_0^3 (x^2 - 3x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3}{2}x^2\right]_0^3$$

$$= \left(9 - 13\tfrac{1}{2}\right) - (0 - 0)$$

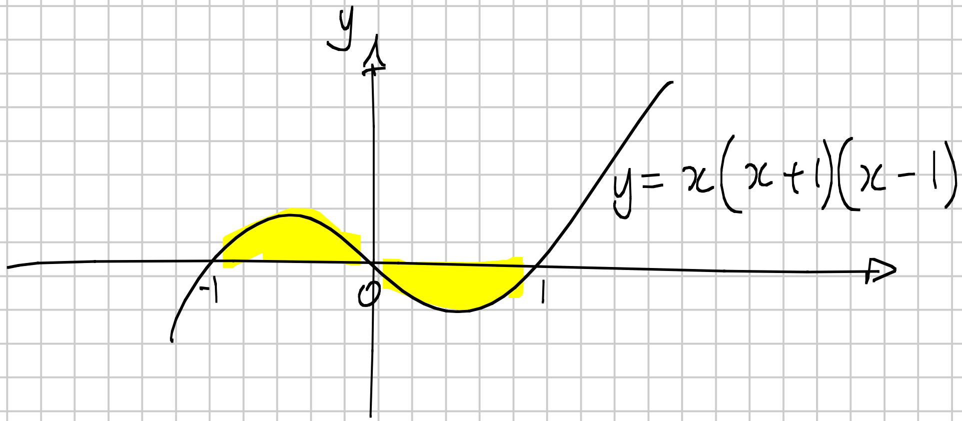
$$= -4\tfrac{1}{2}$$

and this is an area?

So the area is $4\tfrac{1}{2}$.

$$\boxed{\text{Area} = \sum_{i,j} \left| \int_i^j y \, dx \right|}$$

i, j are roots of $y = f(x)$



The wrong way to do it:

$$\int_{-1}^{1} x(x+1)(x-1)\,dx = \int_{-1}^{1} x^3 - x \, dx$$

$$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{1}$$

$$= \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -1/4 - -1/4 = 0.$$

!?

The right way to do it:

1. find the roots
2. integrate between roots
3. make 'em positive $\left\| \int f(x)\,dx \right\|$
4. add 'em up.

1. roots are $-1, 0, 1$

2. $$\int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} (x^3 - x)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0}$$

$$= (0 - 0) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= 1/4$$

$$\int_{0}^{1} f(x)\,dx = \int_{0}^{1} (x^3 - x)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{0}^{1}$$

$$= \left(\frac{1}{4} - \frac{1}{2}\right) - (0 - 0)$$
$$= -\frac{1}{4}$$

③ $\left|\frac{1}{4}\right| = \frac{1}{4}$ $\quad \left|-\frac{1}{4}\right| = \frac{1}{4}$ (make 'em positive)

④ $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $\quad \therefore$ Area $= \frac{1}{2}$.

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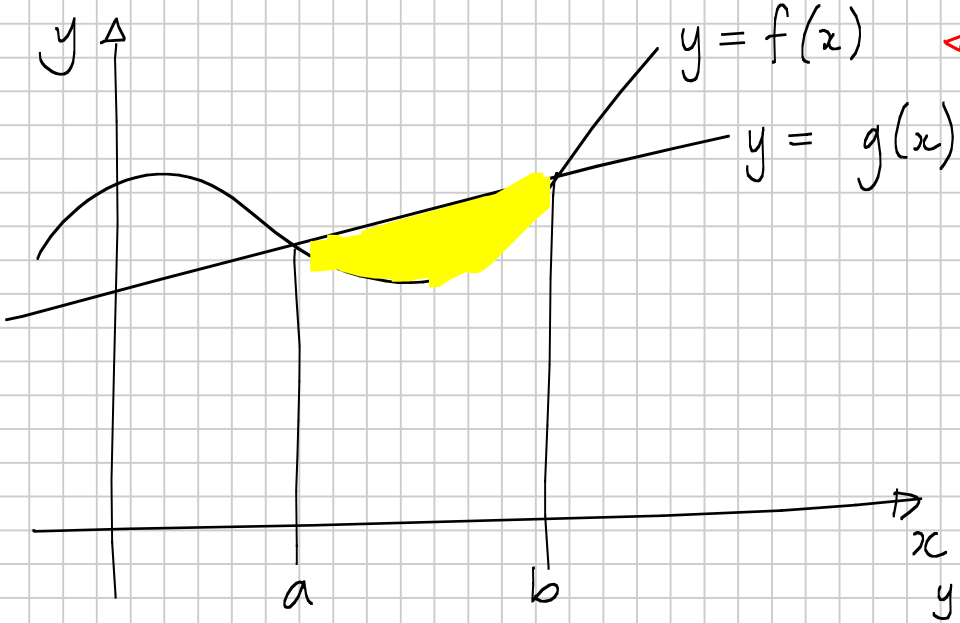
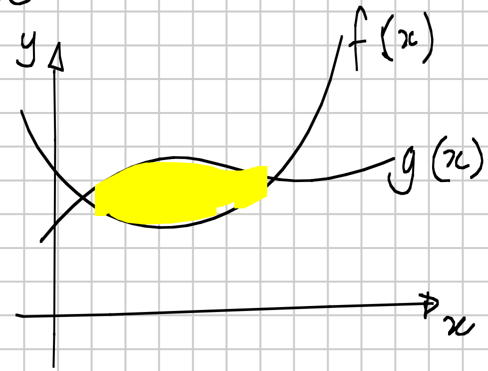
11.4 Area between a curve and a straight line

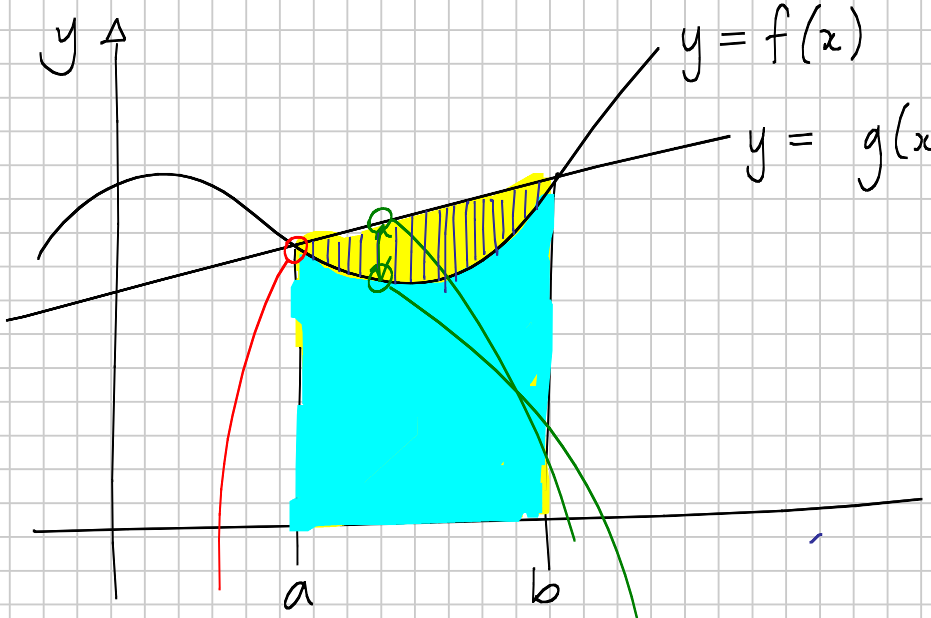


slightly different notation in the book.

where $f(x)$ and $g(x)$ are functions in x.



$$\boxed{\begin{aligned} A &= \int_a^b \big(g(x) - f(x)\big)\,dx \\ &= \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \end{aligned}}$$

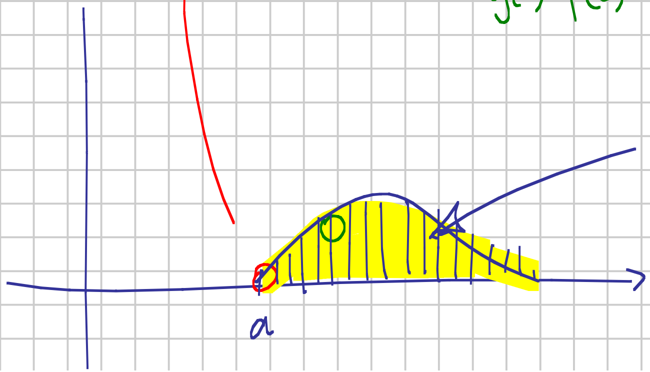


$A_1 = \int_a^b g(x)$

$A_2 =$

$g(x) - f(x) = 0$

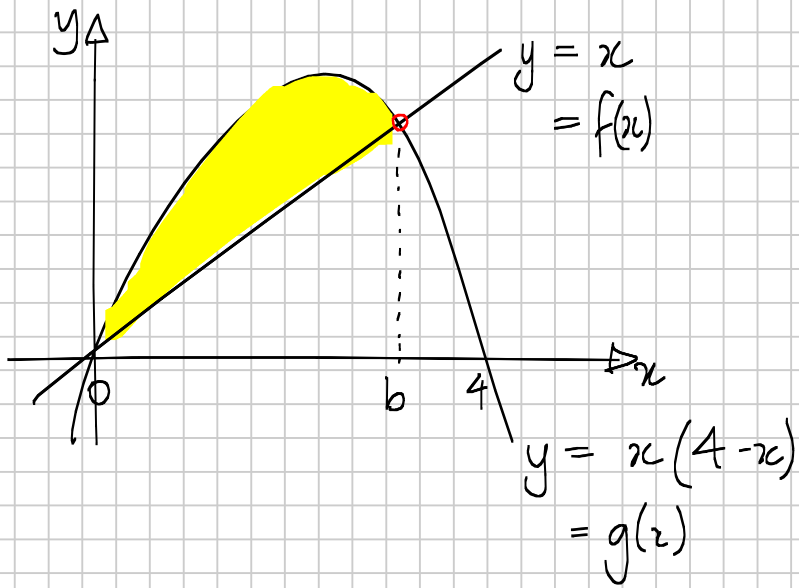
$g(x) - f(x) = \updownarrow$



$\int (g(x) - f(x))\, dx$

Example 6 p164

find the shaded area between $y = x(4-x)$ and $y = x$



① find 'b'

$x = x(4-x)$

$x = 4x - x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0$ or $x = 3$

so $b = 3$

2 Now integrate $\int_a^b \left(g(x) - f(x)\right) dx$

$$= \int_0^3 \left(x(4-x) - x\right) dx$$

$$= \int_0^3 \left(-x^2 + 3x\right) dx$$

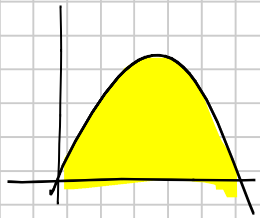
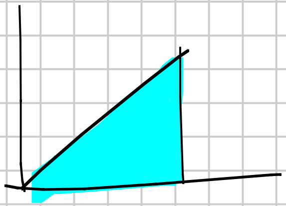
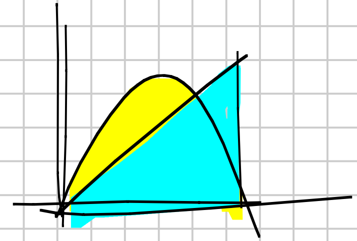
$$= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2\right]_0^3$$

$$= \left(-\frac{1}{3}(3)^3 + \frac{3}{2}(3)^2\right) - \left(-\frac{1}{3}(0)^3 + \frac{3}{2}(0)^2\right)$$

$$= \left(-9 + 13\tfrac{1}{2}\right) - (0)$$

$$= 4\tfrac{1}{2}.$$

Method 2 workout:

 $-$  $=$ 

$$\int_0^3 x(4-x)\,dx - \int_0^3 x\,dx = \text{yellow area}$$

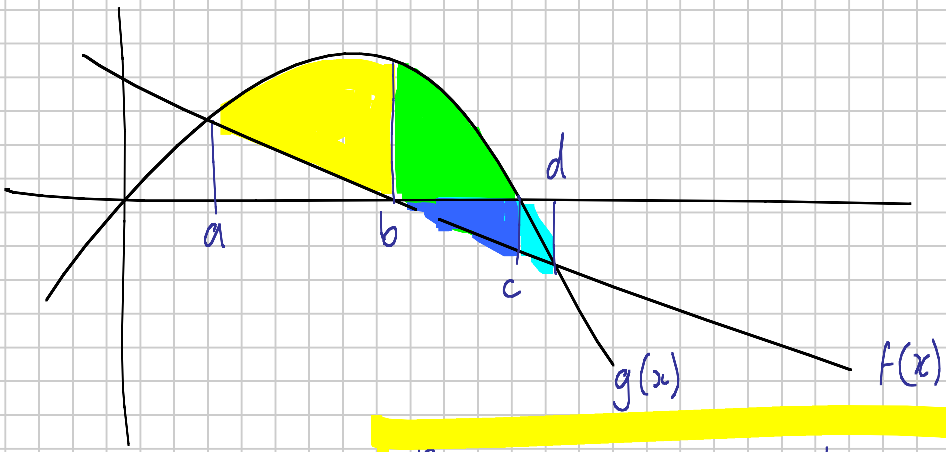
$$= \int_0^3 4x - x^2\,dx - \left[\tfrac{1}{2}x^2\right]_0^3$$

$$= \left[\frac{2\cancel{4}}{\cancel{2}}x^2 - \frac{1}{3}x^3\right]_0^3 - \left(\frac{9}{2} - 0\right)$$

$$= \Big((18-9) - (0-0)\Big) - \left(4\tfrac{1}{2}\right)$$

$$= 9 - 4\tfrac{1}{2} = 4\tfrac{1}{2} \quad (\text{same area !})$$

⚠ Watch out

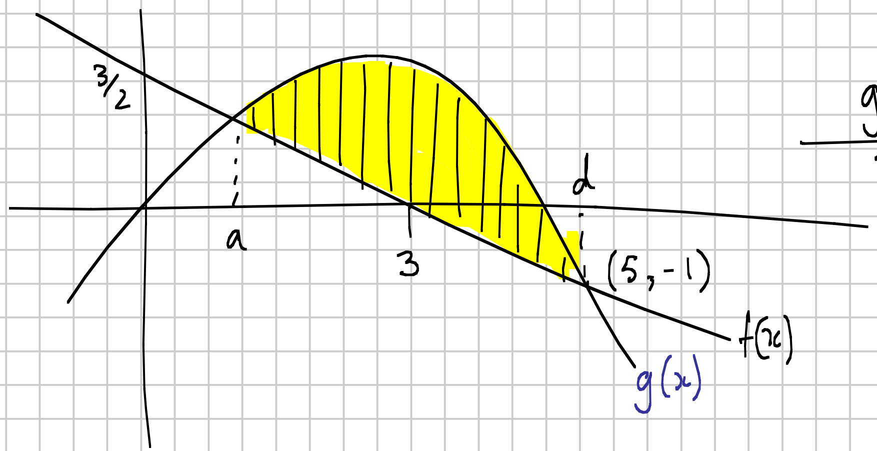


$$\int_a^b g(x)\,dx - \int_a^b f(x)$$

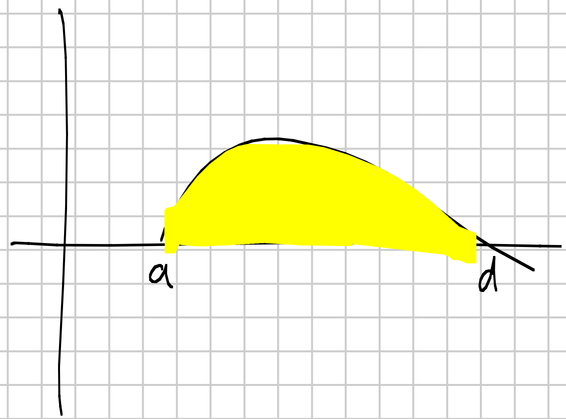
$$\int_b^c g(x)\,dx + \ldots - \int^c f(x)\,dx$$

$$-\left[\int_c^d f(x)\,dx - \int_c^d g(x)\,dx\right]$$

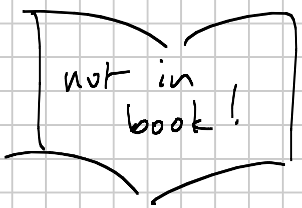
Much easier to do: $\int_a^d \left(g(x) - f(x)\right) dx$



$\xrightarrow[\text{"mapping"}]{g(x) - f(x)}$



example



$$g(x) = x(4 - x)$$

$$f(x) = \frac{3}{2} - \frac{1}{2}x$$

Quick way $\int_a^d \left(g(x) - f(x)\right) dx$

① find 'a' & 'd'

$$\frac{3}{2} - \frac{1}{2}x = x(4-x)$$

$$3 - x = 8x - 2x^2$$

$$2x^2 - 9x + 3 = 0$$

$$x^2 - \frac{9}{2}x + \frac{3}{2} = 0$$

$$\left(x - \frac{9}{4}\right)^2 - \frac{81}{16} + \frac{24}{16} = 0$$

$$\left(x - \frac{9}{4}\right)^2 = \frac{57}{16}$$

$$x = \frac{9 \pm \sqrt{57}}{4}$$

so

$$\int_{\left(\frac{9-\sqrt{57}}{4}\right)}^{\left(\frac{9+\sqrt{57}}{4}\right)} \left(x\left(4-x\right) - \left(\frac{3}{2} - \frac{1}{2}x\right)\right) dx$$

$$= \int_{\frac{9-\sqrt{57}}{4}}^{\frac{9+\sqrt{57}}{4}} \left(-x^2 + \frac{9}{2}x - \frac{3}{2}\right) dx$$

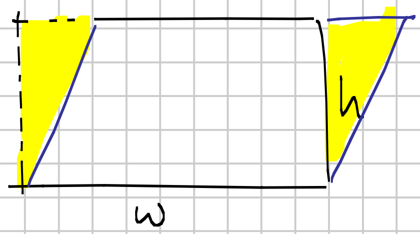
$$= \left[-\frac{1}{3}x^3 + \frac{9}{4}x^2 - \frac{3}{2}x\right]_{\frac{9-\sqrt{57}}{4}}^{\frac{9+\sqrt{57}}{4}}$$

etc...

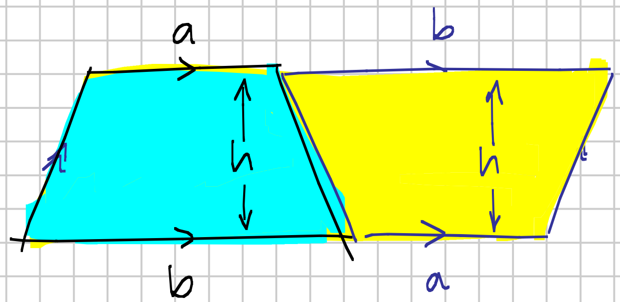
11.5 Trapezium rule

recall that the area of a rectangle is width × height

so the area of a parallelogram is also width × height



Now consider a trapezium...



two identical trapezia make one big parallelogram

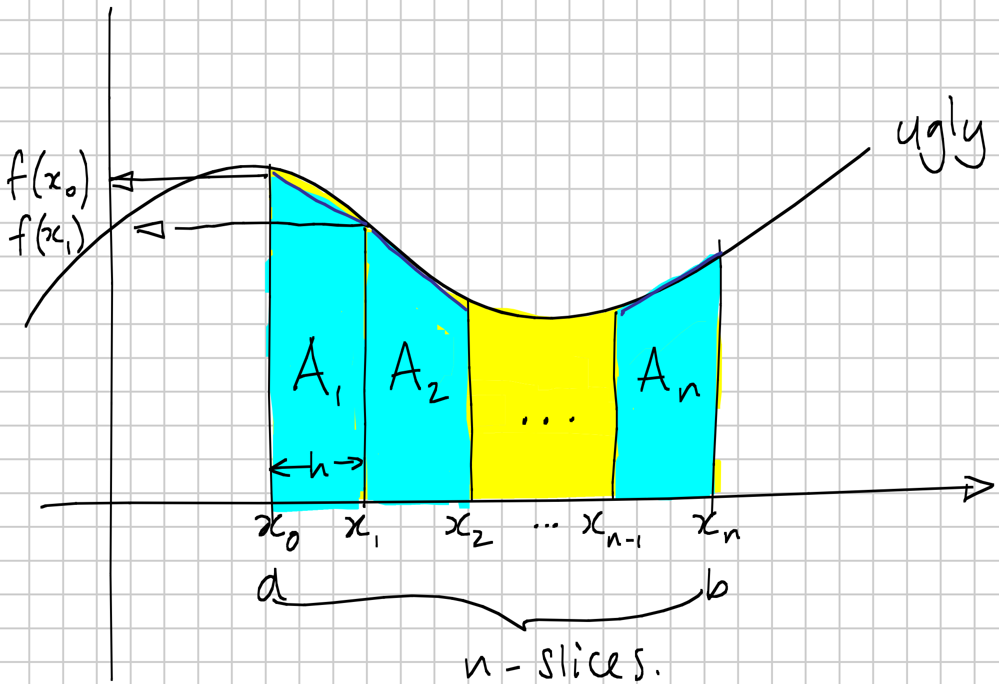
The area of the trapezium is

$\frac{1}{2} \times$ (area of parallelogram)

$= \frac{1}{2} \times$ width $\times$ height

$= \frac{1}{2} \times (a+b) \times h$

$A = \frac{1}{2}(a+b)h$



ugly function $f(x)$ what I can't integrate.

green area is the sum of lots of trapezia

& it's pretty close to the yellow area

now $A_1 = \frac{h}{2}\left(f(x_0) + f(x_1)\right) = \frac{h}{2}(y_0 + y_1)$

$A_2 = \quad . \quad . \quad - \quad . \quad = \frac{h}{2}(y_1 + y_2)$

$$A_n = \; - \; - \; = \frac{h}{2}\left(y_{n-1} + y_n\right)$$

so green area A_T is

$$A_T = A_1 + A_2 + \dots + A_n$$

$$A_T = \frac{h}{2}\left(y_0 + y_1\right) + \frac{h}{2}\left(y_1 + y_2\right) + \frac{h}{2}\left(y_2 + y_3\right) + \dots + \frac{h}{2}\left(y_{n-1} + y_n\right)$$

factorise...

$$A_T = \frac{h}{2}\left(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \dots + y_{n-1} + y_n\right)$$

$$A_T = \frac{h}{2}\left(y_0 + 2\left(y_1 + \dots + y_{n-1}\right) + y_n\right)$$

where $h = \dfrac{b-a}{n}$

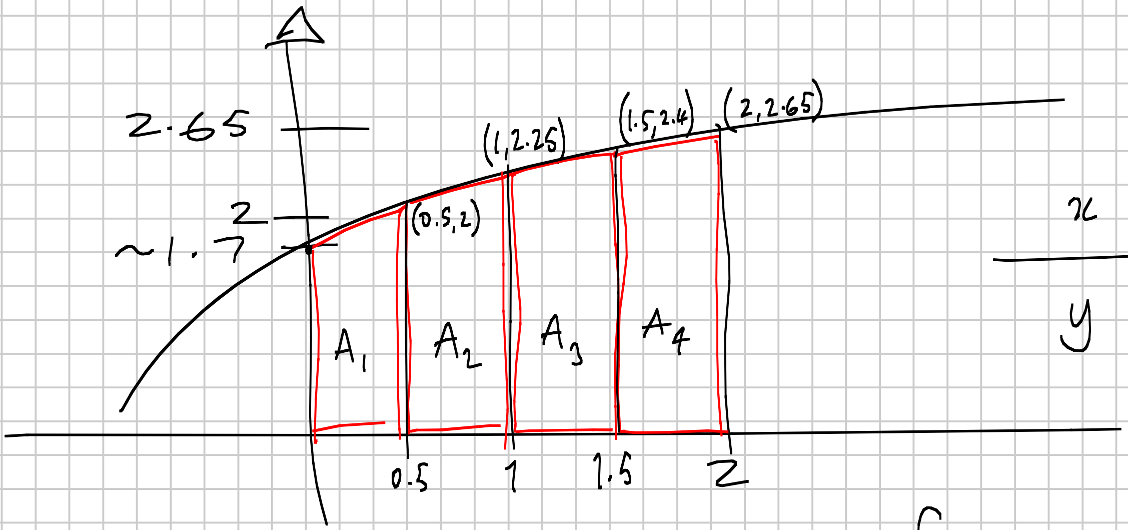
Trapezium Rule.

so $$\boxed{\int_a^b f(x)\,dx \simeq \frac{b-a}{2n}\left(y_0 + 2\left(y_1 + y_2 + \dots + y_{n-1}\right) + y_0\right)}$$

Example 8 p170

$$\int_0^2 \left(\sqrt{2x+3}\right) dx$$ can't do it!

so estimate it using trapezium rule, with four strips

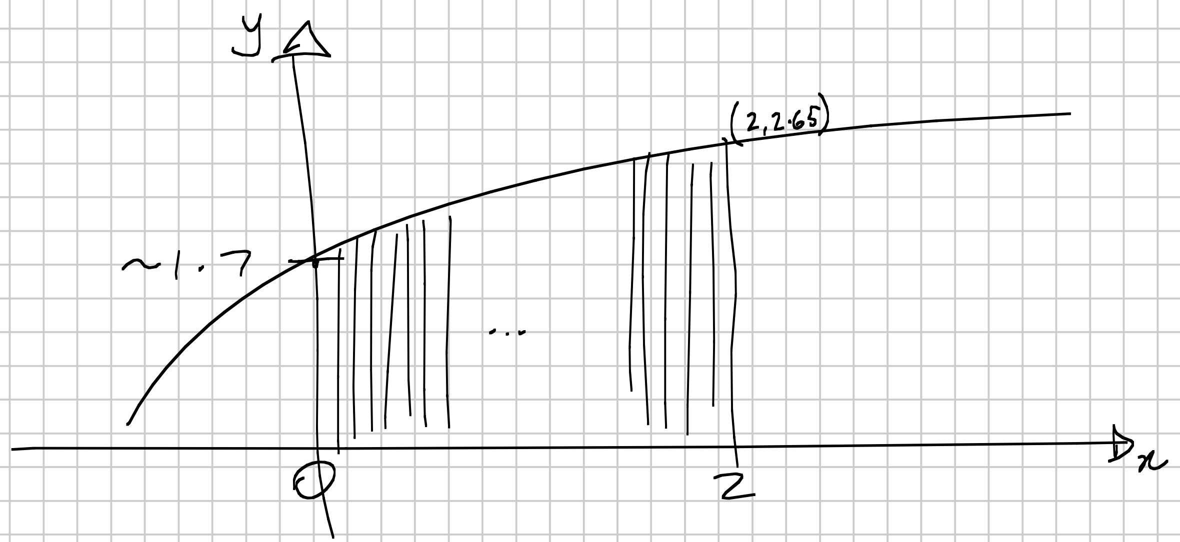


x	0	0.5	1	1.5	2
y	1.732	2.000	2.236	2.449	2.646

width 0.5 into

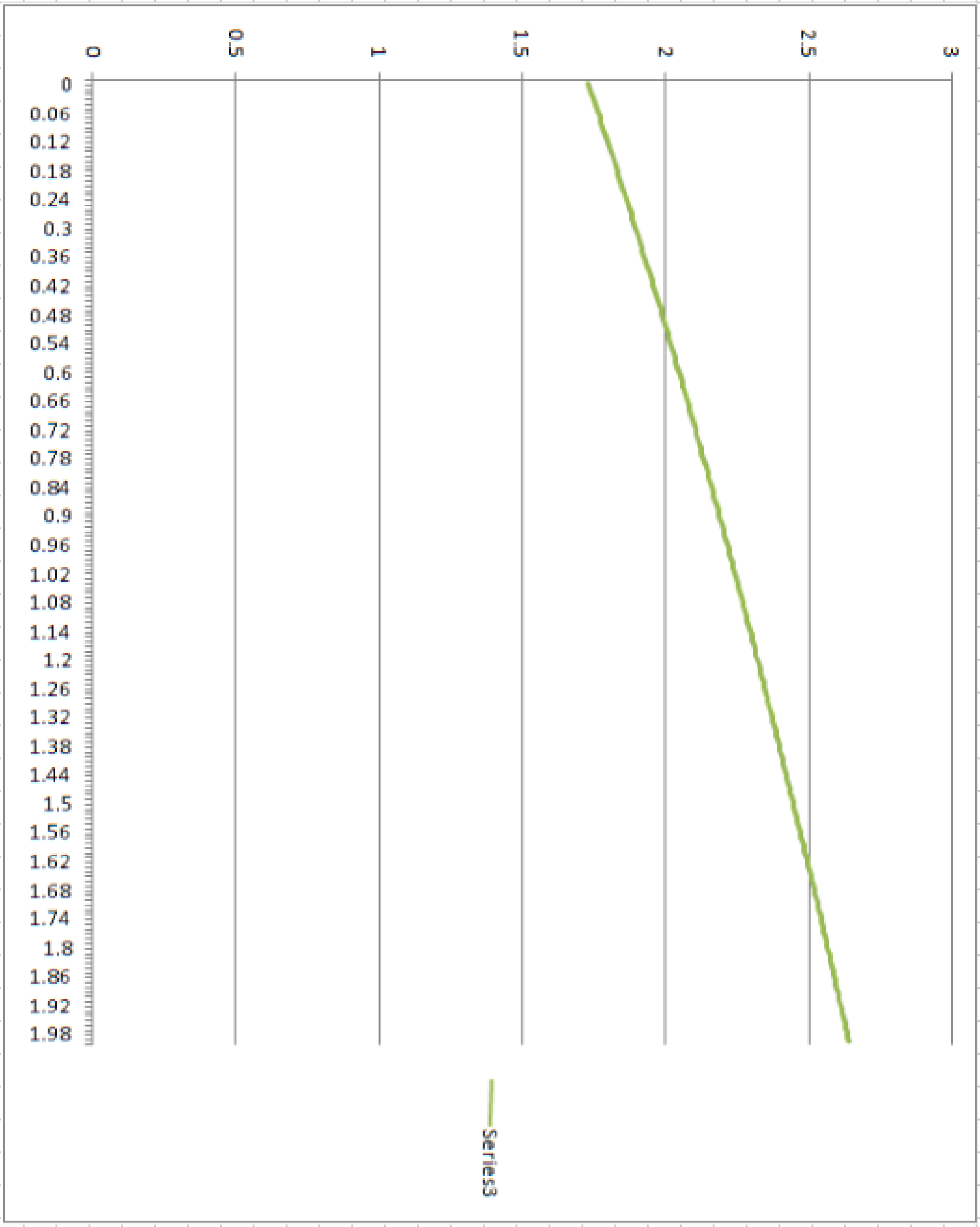
$$A \simeq \frac{0.5}{2}\left[1.732 + 2(2.000 + 2.236 + 2.449) + 2.646\right]$$

$$A \simeq 0.25\,(\text{junk}) = 4.437$$



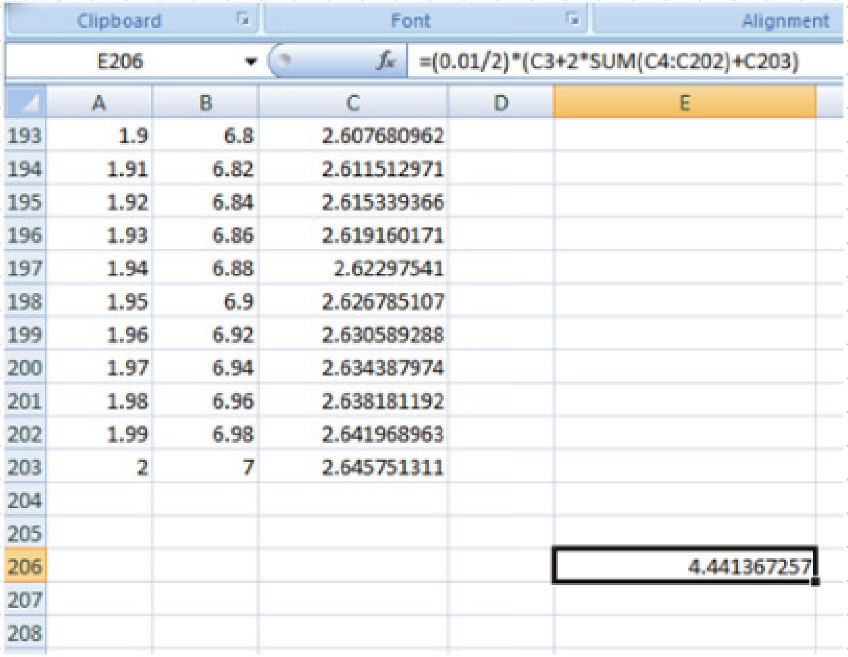
more slices gets a more accurate answer:

see the Excel example below...



0
0.5
1
1.5
2
2.5
3
0
0.06
0.12
0.18
0.24
0.3
0.36
0.42
0.48
0.54
0.6
0.66
0.72
0.78
0.84
0.9
0.96
1.02
1.08
1.14
1.2
1.26
1.32
1.38
1.44
1.5
1.56
1.62
1.68
1.74
1.8
1.86
1.92
1.98
Series3

excel example



Clipboard | Font | Alignment

E206 =(0.01/2)*(C3+2*SUM(C4:C202)+C203)

	A	B	C	D	E
193	1.9	6.8	2.607680962		
194	1.91	6.82	2.611512971		
195	1.92	6.84	2.615339366		
196	1.93	6.86	2.619160171		
197	1.94	6.88	2.62297541		
198	1.95	6.9	2.626785107		
199	1.96	6.92	2.630589288		
200	1.97	6.94	2.634387974		
201	1.98	6.96	2.638181192		
202	1.99	6.98	2.641968963		
203	2	7	2.645751311		
204					
205					
206					4.441367257
207					
208					